As in the isotropic case /1/ we can introduce three $(2n+1)$-harmonic functions that with explicitly express all interconnected functions of (3.13) and (3.14). For this we introduce the $(2n+1, \alpha)$-harmonic function φ_n, the $(2n+1, \gamma)$-harmonic function Ψ_n, and the $(2n+1, \beta)$-harmonic function χ_n for which

$$\Phi_n = \alpha \frac{\partial \varphi_n}{\partial z}, \quad \Psi_n = r^{2n+1} \frac{\partial \varphi_n}{\partial r}, \quad \Phi_n^* = r \frac{\partial \varphi_n}{\partial r} + 2n\varphi_n, \quad \Psi_n^* = -r^{2n}\alpha \frac{\partial \varphi_n}{\partial z}$$

$$P_n = \gamma \frac{\partial \psi_n}{\partial z}, \quad Q_n = r^{2n+1} \frac{\partial \psi_n}{\partial r}, \quad P_n^* = -r \frac{\partial \psi_n}{\partial r} - 2n\psi_n, \quad Q_n^* = r^{2n}\gamma \frac{\partial \psi_n}{\partial z}$$

$$P_{1n} = \beta \frac{\partial \chi_n}{\partial z}, \quad Q_{1n} = r^{2n+1} \frac{\partial \chi_n}{\partial r}, \quad P_{1n} = -r \frac{\partial \chi_n}{\partial r} - 2n\chi_n, \quad Q_{1n} = r^{2n}\beta \frac{\partial \chi_n}{\partial z}$$

The displacements can be expressed in terms of the functions introduced as follows:

$$w_n = r^n \frac{\partial}{\partial z}\left(\frac{2\alpha^2}{\alpha+k}\varphi_n + A\gamma\psi_n\right) \tag{3.15}$$

$$u_n - v_n = r^n \frac{\partial}{\partial r}\left(\frac{2k}{\alpha+k}\varphi_n + B\psi_n + B_1\chi_n\right)$$

$$u_n + v_n = r^{-n} \frac{\partial}{\partial r} r^{2n}\left(\frac{2k}{\alpha+k}\varphi_n + B\psi_n + B_1\chi_n\right)$$

The representations (3.13), (3.14), or (3.15) may be considered as an analog of the Kolosov-Muskhelishvili formulas for the three-dimensional stress state of a transversely isotropic medium.

We note in conclusion that all of the formulas derived remain valid when the roots of Eq.(2.18) are complex. It is only necessary to introduce into consideration (r^k, α)-analytic functions with complex constants α.

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SPECTRAL RELATIONSHIPS FOR THE INTEGRAL OPERATORS GENERATED BY A KERNEL IN THE FORM OF A WEBER-SONIN INTEGRAL, AND THEIR APPLICATION TO CONTACT PROBLEMS *

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Generalized potential theory methods are used to re-establish the spectral relationship /1/ for the integral operators generated by a symmetric kernel in the form of the Weber-Sonin integral in the finite interval $(0, a)$, the kernel containing Jacobi polynomials. Spectral relations are also established for the integral operator generated by the same kernel in the semi-infinite interval (a, ∞), and other allied relationships. The latter

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are used to construct a closed solution of the axisymmetric contact problem of impressing a stamp of annular form in a plane, with an infinite outer radius, into a half-space, the deformation of which obeys a power law. The monographs /2, 3/ give a large number of various spectral relationships in terms of orthogonal polynomials for the integral operators frequently encountered in mathematical physics, and describe a method of orthogonal polynomials based on them. They also show numerous applications of the method to the contact and mixed problems of the theory of elasticity. Spectral relations and their applications to the mixed problems are also given in /4, 5/. These papers are discussed in detail in /6/.

1. Let us determine the eigenvalues and eigenfunctions of the integral operator

$$W\varphi = \int_0^a W_\nu^\gamma(r, \rho)\,\varphi(\rho)\,\rho\,d\rho,\ \varphi(r) \in L_w^2(0, a),\ w(r) > 0$$

generated by a symmetric kernel inthe form of the Weber-Sonin integral

$$W_\nu^\gamma(r, \rho) = \int_0^\infty J_\nu(rt)\,J_\nu(\rho t)\,t^{2\gamma-1}\,dt,\ \operatorname{Re}\nu \geqslant 0,\ \gamma = \mu + \frac{1}{2}, \qquad |\operatorname{Re}\mu| < \frac{1}{2}$$

where $J_\nu(r)$ is the Bessel function of first kind, of order ν. To do this, we consider the following integral equation in a cylindrical system of coordinates (r, ϑ, z):

$$\iint_\omega R^{-2\gamma} p(\rho, \varphi)\,\rho\,d\rho\,d\varphi = f(r, \vartheta) \tag{1.1}$$

and the related generalized potential

$$U(r, \vartheta, z) = \iint_\omega (R^2 + z^2)^{-\gamma}\,p(\rho, \varphi)\,\rho\,d\rho\,d\varphi \tag{1.2}$$

$$(R^2 = r^2 + \rho^2 - 2r\rho\cos(\vartheta - \varphi),\ \omega = \{z = 0; r \leqslant a\},\ |\mu| < {}^1/_2)$$

Putting

$$p(\rho, \varphi) = p_m(\rho)\begin{matrix}\cos m\varphi\\ \sin m\varphi\end{matrix},\quad f(r, \varphi) = f_m(r)\begin{matrix}\cos m\varphi\\ \sin m\varphi\end{matrix},\quad m = 0, 1, 2\ldots$$

we find that instead of (1.1), we can consider the integral equation

$$\int_0^a L_m^\gamma(r, \rho)\,p_m(\rho)\,\rho\,d\rho = f_m(r),\quad m = 0, 1, 2, \ldots \tag{1.3}$$

$$L_m^\gamma(r, \rho) = \int_{-\pi}^{\pi} R_0^{-2\gamma}\cos m\varphi\,d\varphi,\ R_0^2 = r^2 + \rho^2 - 2r\rho\cos\varphi$$

and instead of (1.2), the generalized potential

$$U_m(r, z) = \iint_\omega (R_0^2 + z^2)^{-\gamma}\,p_m(\rho)\cos m\varphi\rho\,d\rho\,d\varphi \tag{1.4}$$

Now, using the results in /7—10/ we can show that the integral equation (1.3) is equivalent to the following boundary value problem for the outside of the circular disc ω:

$$\frac{\partial^2 U_m}{\partial r^2} + \frac{1}{r}\frac{\partial U_m}{\partial r} - \frac{m^2}{r^2}U_m + \frac{\partial^2 U_m}{\partial z^2} + \frac{2\gamma - 1}{z}\frac{\partial U_m}{\partial z} = 0,\ (r, z) \overline{\in} \omega \tag{1.5}$$

$$U_m(r, z)|_{z=0} = f_m(r),\quad 0 < r < a$$

$$U_m(r, z) \backsim (r^2 + z^2)^{-\gamma} P_m,\quad P_m = 0,\quad m = 1, 2, \ldots$$

$$P_0 = 2\pi\int_0^a p_0(\rho)\,\rho\,d\rho,\quad r^2 + z^2 \to \infty$$

Here the density of the sources of finite strength P_m, i.e. the solution of (1.3), is given by

$$-2\pi p_m(r) = \operatorname{sign} z \lim_{z\to 0} |z|^{2\gamma-1}\frac{\partial U_m}{\partial z},\quad r < a \tag{1.6}$$

We construct the solution of (1.5) using the method of separation of variables. To do this we introduce the following coordinates of the oblate spheroid /11/:

$$r = a\operatorname{ch}u\sin v,\ z = a\operatorname{sh}u\cos v;\ 0 \leqslant u < \infty,\ 0 \leqslant v < \pi \tag{1.7}$$

Clearly, the surface $u=0$ represents a doubly covered circular disc ω of radius a. To utilize the results obtained in /11/, we eliminate dU_m/dz from (1.5) by putting

$$U_m(r, z) = |z|^{-\mu} V_m(r, z) \tag{1.8}$$

This yields the differential equation

$$\frac{\partial^2 V_m}{\partial r^2} + \frac{1}{r}\frac{\partial V_m}{\partial r} + \frac{\partial^2 V_m}{\partial z^2} - \frac{m^2}{r^2} V_m + \mu(1-\mu)\frac{V_m}{z^2} = 0, \quad (r, z) \,\overline{\in}\, \omega$$

which, on passing to the coordinates u, v and separating them

$$W_m(u, v) = U_m(u) V_m(v) \tag{1.9}$$
$$(W_m(u, v) = V_m(a \operatorname{ch} u \sin v,\ a \operatorname{sh} u \cos v))$$

leads /11/ to the following ordinary differential equations:

$$\frac{d^2 U_m}{du^2} + \operatorname{th} u \frac{dU_m}{du} - \left[\lambda - \frac{m^2}{\operatorname{ch}^2 u} - \frac{\mu(1-\mu)}{\operatorname{sh}^2 u}\right] U_m = 0, \quad 0 \leqslant u < \infty \tag{1.10}$$

$$\frac{d^2 V_m}{dv^2} + \operatorname{ctg} v \frac{dV_m}{dv} + \left[\lambda - \frac{m^2}{\sin^2 v} + \frac{\mu(1-\mu)}{\cos^2 v}\right] V_m = 0, \quad 0 \leqslant v < \pi \tag{1.11}$$

It is clear that (1.10) transforms to (1.11) on putting $u = i(v - \pi/2)$. We shall therefore only consider (1.11). Further, using the analytic theory of differential equations /12/ we will show that (1.11) can be described by the following Riemann scheme:

$$V_m(v) = P\begin{pmatrix} 0 & 1 & \infty & \\ \mu/2 & m/2 & (1+\sqrt{4\lambda+1})/8 & \cos^2 v \\ (1-\mu)/2 & -m/2 & (1-\sqrt{4\lambda+1})/4 & \end{pmatrix}$$

Therefore ($F(\alpha, \beta; \gamma; x)$ is the Gauss hypergeometric function)

$$V_m(v) = |\cos v|^{\mu} (\sin v)^m F(\alpha, \beta; \gamma; \cos^2 v), \quad 0 \leqslant v < \pi \tag{1.12}$$
$$\begin{matrix}\alpha\\ \beta\end{matrix} = 2m + 2\gamma \pm \sqrt{4\lambda+1}/4$$

Since $\alpha + \beta - \gamma = m \geqslant 0$, the hypergeometric series diverges when $v=0$ and $v=\pi$ /13/. But by virtue of the boundedness of the initial potential $U_m(r, z)$, this function must be bounded for $0 \leqslant v < \pi$. The latter will be ensured provided (see /14/) that the hypergeometric series is truncated, and this will happen when $\alpha = -n$ $(n = 0, 1, 2, \ldots)$. This yields

$$\lambda_n = (m + 2n + \gamma)^2 - 1/4, \quad n = 0, 1, 2, \ldots$$

Further, taking into account the relation given in /15/ connecting the function $F(a, b; c; x)$ with the Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$, we can finally write (1.12) in the form

$$V_m(v) = |\cos v|^{\mu} (\sin v)^m P_n^{(\gamma-1, m)}(1 - 2\cos^2 v), \quad 0 \leqslant v < \pi \tag{1.13}$$

Now the unique solution of (1.10) bounded in the interval $0 \leqslant u < \infty$ and vanishing as $u \to \infty$, will be given by the formula

$$U_m(u) = (\operatorname{sh} u)^{\mu} (\operatorname{ch} u)^m Q_n^{(\gamma-1, m)}(1 + 2\operatorname{sh}^2 u), \quad 0 \leqslant u < \infty \tag{1.14}$$

where $Q_n^{(\alpha,\beta)}(x)$ is the Jacobi function of the second kind /15/. We note that for the boundedness of $U_m(u)$ when $u=0$ it is necessary that $0 \leqslant \mu < 1/2$. However, in accordance with (1.7) and (1.8) we can assume that $|\mu| < 1/2$. This comment also applies to $V_m(v)$ when $v = \pi/2$.

The boundedness of the function $U_m(u)$ for $0 \leqslant u < \infty$ and its vanishing at infinity follow directly from its integral representation in terms of the Jacobi polynomials through the Cauchy-type integral /15/. Using (1.8) and (1.9) we find from (1.13) and (1.14) at once that the boundary value problem (1.5) has a unique normal solution of the form

$$U_m(r, z) = (\operatorname{ch} u \sin v)^m P_n^{(\gamma-1, m)}(1 - 2\cos^2 v)\, Q_n^{(\gamma-1, m)}(1 + 2\operatorname{sh}^2 u) \tag{1.15}$$
$$0 \leqslant u < \infty, \ 0 \leqslant v < \pi, \ n = 0, 1, 2, \ldots$$

where the variables r, z and u, v are connected by the formulas (1.7). Formula (1.6) yields the source density corresponding to the potential (1.15). Using the relations /15/ connecting $Q_n^{(\alpha,\beta)}(x)$ $P_n^{(\alpha,\beta)}(x)$ with $F(a, b; c; x)$ and carrying out certain transformations, we obtain

$$p_m(r) = h_{mn}^{\gamma} r^m (1 - r^2/a^2)^{\gamma-1} P_n^{(\gamma-1,\, m)} (2r^2/a^2 - 1), \quad r < a \tag{1.16}$$
$$(h_{mn}^{\gamma} = a^{2(\gamma-1)-m} \Gamma(\gamma) \Gamma(n + m + 1) [2\pi \Gamma(m + n + \gamma)]^{-1})$$

Next we substitute the value of the potential (1.15) at $u = 0$ and the expression for $p_m(r)$ from (1.16) into (1.3), and take into account (1.4). This yields the spectral relation

$$\int_0^a L_m{}^{\gamma}(r, \rho) P_n^{(m,\, \gamma-1)} (1 - 2\rho^2/a^2) (1 - \rho^2/a^2)^{\gamma-1} \rho^{m+1} d\rho = l_{mn}^{\gamma} r^m P_n^{(m,\, \gamma-1)} (1 - 2r^2/a^2), \quad r < a \tag{1.17}$$

$$l_{mn}^{\gamma} = \pi^2 a^{2(1-\gamma)} \Gamma(m + n + \gamma) \Gamma(n + \gamma) [\sin \pi\gamma \Gamma^2(\gamma) \Gamma(m + n + 1)\, n!]^{-1}$$

Further we express the kernel $L_m{}^{\gamma}(r, \rho)$ from (1.3) in terms of the Weber-Sonin integral. To do this we use the well-known formula /13/, p.92) to express it in terms of a hypergeometric function. After some reduction we find that

$$L_m{}^{\gamma}(r, \rho) = \pi 2^{2(1-\gamma)} \Gamma(1 - \gamma) [\Gamma(\gamma)]^{-1} W_m{}^{\gamma}(r, \rho), \quad r, \rho > 0, \quad m = 0, 1, \ldots \tag{1.18}$$

Since the Weber-Sonin integral $W_\nu{}^{\gamma}(r, \rho)$ is an analytic functionof the parameters ν and γ in the domain of their variation, which ensures that the integral converges, it follows that relation (1.17) can be analytically continued, assuming formally that $m = \nu$, and $\operatorname{Re} \nu \geqslant 0$, $|\operatorname{Re} \mu| < 1/2$. Taking into account the latter and the formula (1.18), we finally obtain, after passing to dimensionless variables $r = ax$, $\rho = ay$, the following spectral relation:

$$\int_0^1 K_\nu{}^{\gamma}(x, y) P_n{}^{\gamma}(y) (1 - y^2)^{\gamma-1} y^{\nu+1} dy = \lambda_{\nu n}^{\gamma} x^{\nu} P_n{}^{\gamma}(x) \tag{1.19}$$

$$K_\nu{}^{\gamma}(x, y) = a^{2\gamma} W_\nu{}^{\gamma}(ax, ay), \quad P_n{}^{\gamma}(x) = P_n^{(\nu,\gamma-1)}(1 - 2x^2)$$

$$\lambda_{\nu n}^{\gamma} = 2^{2(\gamma-1)} \Gamma(n + \nu + \gamma) \Gamma(n + \gamma) [\Gamma(n + \nu + 1)\, n!]^{-1}, \qquad n = 0, 1, 2, \ldots$$

which was obtained by a different method in /1/ (relation (1.19) is identical with (2.17) of /1/ after an obvíous elementary transformation).

We can also obtain an expression related to (1.19) and valid for $x > 1$. Here we note that the surface $v = \pi/2$ represents, according to (1.7), a doubly covered plane annular disc $r \geqslant a$. Therefore we again substitute the value of the potential (1.15) at $v = \pi/2$ and the expression for $p_m(r)$ from (1.16) into (1.3), and take (1.4) into account. Repeating the arguments used in deriving (1.19) we arrive, after certain transformations, at the following relation:

$$\int_0^1 K_\nu{}^{\gamma}(x, y) P_n{}^{\gamma}(y) (1 - y^2)^{\gamma-1} y^{\nu+1} dy = \omega_{\nu n}^{\gamma} x^{\nu} Q_n^{(\gamma-1,\, \nu)} (2x^2 - 1), \quad x > 1$$

$$\omega_{\nu n}{}^{\gamma} = (-1)^n 2^{2\gamma-1} \sin \pi\gamma \Gamma(n + \nu + \gamma) \Gamma(n + \gamma) [\pi \Gamma(n + \nu + 1)\, n!]^{-1}, \quad n = 0, 1, 2, \ldots$$

2. Let us now derive the spectral relations for the integral operator generated by a symmetric kernel in the form of a Weber-Sonin integral in the semi-infinite interval (a, ∞). In this connection we shall regard ω everywhere in (1.1)-(1.4) as an annular disc $r \geqslant a$ of the plane $z = 0$, and replace the interval $(0, a)$ by (a, ∞). We again assume that the density of the sources giving rise to the generalized potential $U(r, \vartheta, z)$ is finite, i.e.

$$P = \iint_{\omega} p(\rho, \varphi) \rho\, d\rho\, d\varphi < \infty$$

although this condition need not hold for separate harmonics.

In the present case we again arrive at a boundary value problem of the type (1.5), and to construct its solution we again use (1.7) to introduce the coordinates of the oblate spheroid. In this case we must however assume that $-\infty < u < \infty$, $0 \leqslant v \leqslant \pi/2$. The surface $v = \pi/2$ represents a doubly covered annular disc $\omega = \{z = 0; r \geqslant a\}$.

After the separation of variables we obtain, in these coordinates, the same differential equations (1.10) and (1.11) in which the separation λ is replaced by $-\lambda^2 - 1/4$. But now the equation (1.10) will be considered in the interval $-\infty < u < \infty$ and (1.11) in the segment $0 \leqslant v \leqslant \pi/2$.

The differential equation (1.1) has the following two linearly independent solutions:

$$(\cos v)^{\mu} (\sin v)^m F(\alpha, \bar{\alpha}; \gamma; \cos^2 v)$$
$$(\cos v)^{1-\mu} (\sin v)^m F(\alpha - \gamma + 1, \bar{\alpha} - \gamma + 1; 2 - \gamma; \cos^2 v)$$
$$\alpha = (m + \gamma + i\lambda)/2, \quad \lambda > 0, \quad 0 \leqslant v \leqslant \pi/2, \quad \gamma = \mu + 1/2$$

Using the well-known formula /13/ we can write

$$F(\alpha, \bar{\alpha}; \gamma; \cos^2 v) = (\sin v)^{-2m} F(\gamma - \alpha, \gamma - \bar{\alpha}; \gamma; \cos^2 v) \tag{2.1}$$
$$F(\alpha - \gamma + 1, \bar{\alpha} - \gamma + 1; 2 - \gamma; \cos^2 v) = (\sin v)^{-2m} F(1 - \alpha, \quad 1 - \bar{\alpha}; 2 - \gamma; \cos^2 v),$$

from which it follows that the functions are of order $(\sin v)^{-2m}$ as $v \to 0$, i.e. the hypergeometric functions shown have, for the given values of the parameters, a singularity at the point $v = 0$. Using this, we write the solution of (1.11) in the form $(0 \leqslant v \leqslant \pi/2)$.

$$V_m(v) = (\cos v)^{\mu} (\sin v)^m [\varphi_m^{\gamma}(\lambda, \sin v) - \varkappa_m^{\gamma}(\lambda)(\cos v)^{1-2\mu} \times \psi_m^{\gamma}(\lambda, \sin v)] \tag{2.2}$$
$$\varphi_m^{\gamma}(\lambda, x) = F(\alpha, \bar{\alpha}; \gamma; 1 - x^2), \quad \psi_m^{\gamma}(\lambda, x) = F(\alpha - \gamma + 1, \quad \bar{\alpha} - \gamma + 1; 2 - \gamma; 1 - x^2)$$

and choose the unknown function $\varkappa_m^{\gamma}(\lambda)$ in such a manner, that the function $(\cos v)^{-\mu} V_m(v)$ is bounded on the segment $0 \leqslant v \leqslant \pi/2$. Since $\gamma - \alpha - \bar{\alpha} + m = 0$, using (2.1) and the formulas for analytic continuation of a hypergeometric function /13/, we find

$$\varkappa_m^{\gamma}(\lambda) = |\Gamma[(m - \gamma + 2 + i\lambda)/2]|^2 \Gamma(\gamma) |\Gamma[(m + \gamma + i\lambda)/2]|^{-2} [\Gamma(2 - \gamma)]^{-1} \tag{2.3}$$

Thus the required solution of (1.11) is given in this case by the formulas (2.2)−(2.3).

Let us now consider (1.10). We find at once that it has, in the case in question, two linearly independent solutions:

$$U_m(u) = |\operatorname{sh} u|^{\mu} (\operatorname{ch} u)^m \varphi_m^{\gamma}(\lambda, \operatorname{ch} u), \quad -\infty < u < \infty \tag{2.4}$$
$$U_m(u) = |\operatorname{sh} u|^{1-\mu} (\operatorname{ch} u)^m \psi_m^{\gamma}(\lambda, \operatorname{ch} u), \quad m = 0, 1, 2, \ldots$$

Here the functions $|\operatorname{sh} u|^{-\mu} U_m(u)$ are bounded for $-\infty < u < \infty$ and vanish exponentially in accordance with the asymptotic formulas /13/ as $|u| \to \infty$. Consequently, taking (2.2) and (2.4) into account we find that the boundary value problem (1.5) has, in this case, the normal solutions

$$U_m(r, z) = (\operatorname{ch} u \sin v)^m \chi_m^{\gamma}(\lambda, \sin v) \begin{matrix} \varphi_m^{\gamma}(\lambda, \operatorname{ch} u) \\ |\operatorname{sh} u|^{1-2\mu} \psi_m^{\gamma}(\lambda, \operatorname{ch} u) \end{matrix} \tag{2.5}$$
$$\chi_m^{\gamma}(\lambda, \sin v) = \varphi_m^{\gamma}(\lambda, \sin v) - \varkappa_m^{\gamma}(\lambda)(\cos v)^{1-2\mu} \psi_m^{\gamma}(\lambda, \sin v)$$
$$-\infty < u < \infty, \quad 0 \leqslant v \leqslant \pi/2, \quad m = 0, 1, 2, \ldots$$

where the variables r, z and u, v are again connected by (1.7).

Let us calculate the source intensities corresponding to the potentials (2.5). Noting that

$$\left.\frac{\partial U_m}{\partial z}\right|_{z=0} = -(a|\operatorname{sh} u|)^{-1} \left.\frac{\partial U_m}{\partial v}\right|_{v=\pi/2}, \quad -\infty < u < \infty$$

we use (1.6) to obtain from (2.5) $(r = a \operatorname{ch} u, \ 0 \leqslant u < \infty)$

$$p_m(r) = \pi^{-1}(1 - \gamma) a^{2(\gamma-1)} \varkappa_m^{\gamma}(\lambda) (\operatorname{ch} u)^m \begin{matrix} (\operatorname{sh} u)^{2(\gamma-1)} \varphi_m^{\gamma}(\lambda, \operatorname{ch} u) \\ \psi_m^{\gamma}(\lambda, \operatorname{ch} u) \end{matrix} \tag{2.6}$$

Next, taking into account (1.4) we substitute the expression for $p_m(r)$ given in (2.6) and the values of the corresponding potentials into (1.3), where we replace the interval $(0, a)$ by (a, ∞). As a result we arrive at the following spectral relations:

$$\int_a^{\infty} L_m^{\gamma}(r, \rho) \varphi_m^{\gamma}(\lambda, \rho/a) (\rho^2/a^2 - 1)^{\gamma-1} \rho^{m+1} d\rho = \sigma_m^{\gamma}(\lambda) r^m \varphi_m^{\gamma}(\lambda, r/a), \quad r > a \tag{2.7}$$
$$\int_a^{\infty} L_m^{\gamma}(r, \rho) \psi_m^{\gamma}(\lambda, \rho/a) \rho^{m+1} d\rho = \sigma_m^{\gamma}(\lambda) r^m (r^2/a^2 - 1)^{1-\gamma} \psi_m^{\gamma}(\lambda, r/a)$$
$$\sigma_m^{\gamma}(\lambda) = \pi [(1 - \gamma) \varkappa_m^{\gamma}(\lambda)]^{-1} a^{2(1-\gamma)}, \quad m = 0, 1, 2, \ldots$$

In the special case when $\mu = 0$, using the relation given in (/13/, p.130), we find ($P_\delta^{-m}(x)$ is the Legendre function of first kind)

$$\varphi_m^{1/2}(\lambda, x) = 2^m \pi^{-1/2} (ix)^{-m} |\Gamma[(m + 3/2 + i\lambda)/2]|^2 \varphi_m(\lambda, x), \quad x > 1 \tag{2.8}$$
$$\varphi_m^{1/2}(\lambda, x) = -2^{m-1} \pi^{-1/2} (ix)^{-m} (x^2 - 1)^{-1/2} |\Gamma[(m + 1/2 + i\lambda) | 2]|^2 \psi_m(\lambda, x)$$
$$\varphi_m(\lambda, x) = [P_\delta^{-m}(i\sqrt{x^2 - 1}) + P_\delta^{-m}(-i\sqrt{x^2 - 1})]/2, \quad \delta = -1/2 + i\lambda$$
$$\psi_m(\lambda, x) = [P_\delta^{-m}(i\sqrt{x^2 - 1}) - P_\delta^{-m}(-i\sqrt{x^2 - 1})]/(2i)$$

Further, remembering what was said above about the analytical properties of the Weber-Sonin integral and changing to dimensionless coordinates by means of the formula $r = ax$, $\rho = ay$, we finally obtain in place of (2.7) the following spectral relation:

$$\int_1^\infty K_\nu^\gamma(x, y)\,\varphi_\nu^\gamma(\lambda, y)\,(y^2 - 1)^{\gamma-1}\,y^{\nu+1}\,dy = \rho_\nu^\gamma(\lambda)\,x^\nu\varphi_\nu^\gamma(\lambda, x), \quad x > 1 \tag{2.9}$$

$$\int_1^\infty K_\nu^\gamma(x, y)\,\psi_\nu^\gamma(\lambda, y)\,y^{\nu+1}\,dy = \rho_\nu^\gamma(\lambda)\,x^\nu(x^2 - 1)^{1-\gamma}\,\psi_\nu^\gamma(\lambda, x)$$

$$\rho_\nu^\gamma(\lambda) = 2^{2(\gamma-1)}\,|\,\Gamma\,[(\nu + \gamma + i\lambda)/2]|^2\;|\,\Gamma[(\nu - \gamma + 2 + i\lambda)/2]|^{-2}$$

where the expressions for $\varphi_\nu^\gamma(\lambda, x)$ and $\psi_\nu^\gamma(\lambda, x)$ are obtained from (2.2) and (2.8) by formal replacement of m by ν.

Relations (2.9) yield generalized eigenfunctions of the integral operator generated by the symmetric kernel, in the form of a Weber-Sonin integral in the semi-infinite interval $(1, \infty)$.

To obtain relations allied to (2.9) and valid for $0 < x < 1$, we put $u = 0$ in (2.5), use (1.4) and (2.6) and formally replace m by ν. After certain transformations we obtain

$$\int_1^\infty K_\nu^\gamma(x, y)\,\varphi_\nu^\gamma(\lambda, y)\,(y^2 - 1)^{\gamma-1}\,y^{\nu+1}\,dy = \tag{2.10}$$
$$\rho_\nu^\gamma(\lambda)\,x^\nu\,[\varphi_\nu^\gamma(\lambda, x) - \varkappa_\nu^\gamma(\lambda)\,(1 - x^2)^{1-\gamma}\,\psi_\nu^\gamma(\lambda, x)], \quad 0 < x < 1$$

$$\int_1^\infty K_\nu^\gamma(x, y)\,\psi_\nu^\gamma(\lambda, y)\,y^{\nu+1}\,dy = 0$$

where $\varkappa_\nu^\gamma(\lambda)$ is obtained from (2.3) after replacing m by ν, the remaining notation being unchanged.

In the special case when $\mu = 0$, using (2.8), in which m has been replaced by ν, relations (2.9) and (2.10) become, respectively,

$$K_\nu\varphi_\nu = \rho_\nu^{1/2}(\lambda)\,\varphi_\nu, \quad K_\nu\psi_\nu = \rho_\nu^{1/2}(\lambda)\,\psi_\nu; \quad x > 1$$
$$K_\nu\varphi_\nu = \rho_\nu^{1/2}(\lambda)\,\chi_\nu, \quad K_\nu\psi_\nu = 0; \quad 0 < x < 1$$
$$K_\nu\varphi = \int_1^\infty K_\nu^{1/2}(x, y)\,\varphi(y)\,(y^2 - 1)^{-1/2}\,y\,dy, \quad 0 < x < \infty$$
$$\begin{matrix}\varphi_\nu \\ \nu\end{matrix} = \begin{matrix}\varphi_\nu(\lambda, x) \\ \psi_\nu(\lambda, x)\end{matrix} \quad (x > 1)$$
$$\chi_\nu = \chi_\nu(\lambda, x) = P_\delta^{-\nu}\left(\sqrt{1 - x^2}\right), \quad (0 < x < 1)$$

The formulas for expanding arbitrary functions of a fairly general class over the families of functions $\varphi_\nu^\gamma(\lambda, x)$ and $\psi_\nu^\gamma(\lambda, x)$, can be obtained by the well known method described in /16/ by considering the hypergeometric differential equation in a semi-infinite interval. For the given values of the parameters we have, at the left end of this interval, the case of the Weyl limit circle. Other cases are discussed in /16, 17/.

Without bothering to discuss the details of the proposed method, we give the final result: the formulas for expanding an arbitrary function $f(x)$ over the family of functions $\varphi_\nu^\gamma(\lambda, x)$ have the following form:

$$F(\lambda) = \int_1^\infty \varphi_\nu^\gamma(\lambda, x)\,(x^2 - 1)^{\gamma-1}\,x^{2\nu+1} f(x)\,dx \tag{2.11}$$

$$f(x) = \int_0^\infty \varphi_\nu^\gamma(\lambda, x)\,F(\lambda)\,h(\lambda)\,d\lambda$$

$$h(\lambda) = [2\pi^2\Gamma^2(\gamma)]^{-1}\,\lambda\,\operatorname{sh}\pi\lambda\,|\,\Gamma[(\nu + \gamma + i\lambda)/2]\,\Gamma\,[(\gamma - \nu + i\lambda)/2]\,|^2$$

and the formulas for the family of functions $\psi_\nu^\gamma(\lambda, x)$ are exactly similar. Using (2.8) for $m = 0$ we can show that (2.11) transposes into the well-known formulas given in /18/. Then, taking (2.11) into account, we can use the first spectral relation of (2.9) to obtain, for the kernel $(xy)^{-\nu}K_\nu^\gamma(x, y)$, a bilinear expansion into an integral.

3. We will illustrate the results obtained by considering the axisymmetric contact problem of impressing a stamp which has in the plane the form of a circular annulus with an infinite outer radius, into the half-space $z < 0$. We shall assume that the material of the half-space obeys a power law $\sigma_i = K_0\varepsilon_i^q$ $(0 < q \leqslant 1)$ of the non-linear theory of steady creep /19, 20/ where σ_i and ε_i are, respectively, the stress and strain rate intensities, while K_0

and q are physical constants. Adhering to the generalized principle of superposition of displacements /19, 20/, we can reduce the solution of the problem in question to the solution of the integral equation

$$\int_a^\infty L_0^\gamma(r,\rho)\,p_0(\rho)\,\rho\,d\rho = A^{2(\gamma-1)}[\delta - f_0(r)]^{2(1-\gamma)} \tag{3.1}$$

$$2\pi\int_a^\infty p_0(\rho)\,\rho\,d\rho = P_0 < \infty,\quad \mu = (1-q)/2$$

where $p_0(r)$ is the contact pressure, P_0 is the resultant of the external forces applied to the stamp, δ is the settling of the stamp, $f_0(r)$ is a function describing its base, A is a constant expressed in terms of K_0 and q, and the remaining notation is as before.

Introducing the dimensionless variables

$$r = ax,\ \rho = ay,\ p_0(r) = A^{2(\gamma-1)}\varphi(x)$$
$$f(x) = 2^{2(\gamma-1)}\Gamma(\gamma)\,[\pi\Gamma(1-\gamma)]^{-1}\,[\delta_0 - a^{-1}f_0(ax)]^{2(1-\gamma)},\ \delta_0 = \delta/a$$

we can reduce (3.1) to

$$\int_1^\infty K_0^\gamma(x,y)\,\varphi(y)\,y\,dy = f(x) \tag{3.2}$$

and write the solution of (3.2) in the form

$$\varphi(x) = \omega(x)\int_0^\infty \varphi_0^\gamma(\lambda,x)\,\Phi(\lambda)\,d\lambda,\quad \omega(x) = (x^2-1)^{\gamma-1},\quad x>1 \tag{3.3}$$

Substituting (3.3) into (3.2) and using (2.9), (3.11), we obtain

$$\Phi(\lambda) = \frac{2^{2(1-\gamma)}\lambda\,\mathrm{sh}(\pi\lambda)}{\Gamma^2(\gamma)\,[\mathrm{ch}(\pi\lambda)-\cos\pi\gamma]}\int_1^\infty \varphi_0^\gamma(\lambda,y)\,(y^2-1)^{\gamma-1}\,y f(y)\,dy \tag{3.4}$$

The reduced vertical displacements outside the stamp are given by the formula $(0<x<1)$

$$w_0(x) = \int_1^\infty K_0^\gamma(x,y)\,\varphi(y)\,y\,dy,\ w_0(x) = \frac{2^{2(\gamma-1)}\Gamma(\gamma)}{\pi\Gamma(1-\gamma)}\left[-\frac{u_z(ax)}{a}\right]^{2(1-\gamma)}$$

where $u_z(r)$ are the true displacements of the foundation outside the stamp. Using (2.10) we find at once $(0<x<1)$

$$w_0(x) = \int_0^\infty [\varphi_0^\gamma(\lambda,x) - \varkappa_0^\gamma(\lambda)(1-x^2)^{1-\gamma}\psi_0^\gamma(\lambda,x)]\,\Phi(\lambda)\,\rho_0^\gamma(\lambda)\,d\lambda \tag{3.5}$$

In the case of a linearly elastic half-space, when $\gamma = {}^1/_2$, formulas (3.3)–(3.5), taking (2.8) into account, yield

$$\varphi(x) = 8\pi^3(x^2-1)^{-1/2}\int_0^\infty \varphi_0(\lambda,x)\,\lambda\,\mathrm{th}(\pi\lambda)\times\{\mathrm{ch}(\pi\lambda)\,|\Gamma[(\delta+1)/2]|^2\}^{-2}\,\Phi(\lambda)\,d\lambda$$

$$\Phi(\lambda) = \pi^{-1}\int_1^\infty \varphi_0(\lambda,x)\,(x^2-1)^{-1/2}x f(x)\,dx,\quad \delta = -{}^1/_2 + i\lambda,\quad x>1$$

$$w_0(x) = \left(2\sqrt{\pi}\right)^{-1}\int_0^\infty P_\delta\left(\sqrt{1-x^2}\right)|\Gamma[(\delta+1)/2]|^2\,\Phi(\lambda)\,d\lambda,\ 0<x<1$$

Finally, to find δ_0 we equate the asymptotics of the left and right sides of (3.2) as $x\to\infty$. Hence we obtain that the following asymptotic expression must hold:

$$a^{-1}f_0(ax) \sim \delta_0 - Q^s x^{1-2c},\ x\to\infty$$
$$Q = P_0A^{2(1-\gamma)}/a^2,\ s = [2(1-\gamma)]^{-1},\ c = 1/q$$

and this in fact yields δ_0.

Note that the results of Sect.3 can be extended to the problem in question using the formulation of the linear theory of elasticity /6–9/ when the modulus of elasticity of the half-space has a power-low variation with depth.

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TRANSONIC FLOW OF AN ELASTIC MEDIUM PAST A THIN SOLID*

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A plane problem of the steady state of a body in an infinite elastic medium in the range of sonic velocities is considered. The generalized Hilbert problem arises for the complex function determining the longitudinal part of the velocity and stress field, and the transverse part of the field is expressed simply by the solution of the Hilbert problem. The separation of the medium from the body contour at the trailing edge is computed. In the former case the position of the separation point is not known, and the method of fixing this point differs from that in /1/ where the problem of wedging is considered at sub-Rayleigh velocities. In /1/ the free surface is formed before the frontal part of the wedge and the separation point is found from the condition that the stresses are finite. In the present problem, just as in the case of super-Rayleigh subsonic motion of a wedge /2, 3/, the condition that the stresses are finite (and even continuous) at the separation point is ensured by the solution beforehand, and a more accurate analysis is required, which will include, to clarify the problem, the computation of the first few terms of the asymptotic expansion of the solution near the separation point. The separation point is fixed using the condition of attachment of the flow in the zone of contact, and the condition of impermeability of the region between the separation point and the trailing edge of the body. The demand that both these physical conditions are met locally near the point of

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